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Finite Integrals and Fourier Series Involving General Polynomial, Multivariable Mittage-Leffler Function and Modified *I*-Function of Two Variables

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Abstract :

In literature, a lot of remarkable finite and infinite integral formulas and fourier series involving various special functions have been given. In this paper, first we evaluate three finite integrals which involve Srivastava's polynomials, multivariable Mittage-Leffler function and modified I-function of two variables. In the next section, with the help of these integrals we shall obtain three fourier series. Our results are believed to be new in the literature. On account of general nature of our findings, a large number of new and known results follow as special cases of our findings.

Keywords and Phrases : Srivastava polynomial, multivariable Mittage-Leffler function, modified *I*-function, finite integral formulas.

2010 Mathematics Subject Classification : Primary : 33C60, 33C99; Secondary : 44A20

1. Introduction and Preliminaries :

In 1979, Prasad and Prasad [8] introduced modified H-function of two variables and in 2012, Shantha Kumari et al. [3] defined I-function of two variables. In this paper, we are defining modified I-function of two variables, which is the generalization of both the modified H-function of two variables of Prasad and Prasad and I-function of two variables Shantha Kumari, in the following manner :

$$I[z_1, z_r] = I^{m,n:m_1,n_1:m_2,n_2:m_3,n_3}_{p,q:p_1,q_1:p_2,q_2:p_3,q_3}$$

$$\left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (a_j;\alpha_j, A_j;\xi_j)_{1,p} : (c_j;\gamma_j, C_j;\xi'_j)_{1,p_1} : (e_j, E_j;U_j)_{1,p_2} ; (g_j, G_j;P_j)_{1,p_3} \\ (b_j;\beta_j, B_j;\eta_j)_{1,q} : (d_j;\delta_j, D_j;\eta'_j)_{1,q_1} : (f_j, F_j;V_j)_{1,q_2} ; (h_j, H_j;Q_j)_{1,q_3} \end{matrix} \right]$$

$$= \frac{1}{(2\pi i)^2} \int_{L1} \int_{L2} \psi(s_1, s_2)\, \theta_1(s_1)\, \theta_2(s_2)\, z_1^{s_1} z_2^{s_2}\, ds_1\, ds_2 \tag{1}$$

where

$$\psi(s_1, s_2) = \frac{\prod_{j=1}^{m} \Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2) \prod_{j=1}^{n} \Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 + A_j s_2)}{\prod_{j=1}^{q} \Gamma^{\eta_j}(1 - b_j + \beta_j s_1 + B_j s_2) \prod_{j=n+1}^{p} \Gamma^{\xi_j}(a_j - \alpha_j s_1 - A_j s_2)}$$

$$\times \frac{\prod_{j=1}^{m_1} \Gamma^{\eta'_j}(d_j - \delta_j s_1 + D_j s_2) \prod_{j=1}^{n_1} \Gamma^{\xi'_j}(1 - c_j + \gamma_j s_1 - C_j s_2)}{\prod_{j=1}^{q_1} \Gamma^{\eta'_j}(1 - d_j + \delta_j s_1 - D_j s_2) \prod_{j=n+1}^{p_1} \Gamma^{\xi'_j}(c_j - \gamma_j s_1 + C_j s_2)} \tag{2}$$

$$\theta_1(s_1) = \frac{\prod_{j=1}^{m_2} \Gamma^{U_j}(f_j - F_j s_1) \prod_{j=1}^{n_2} \Gamma^{V_j}(1 - e_j + E_j s_1)}{\prod_{j=m_2+1}^{q_2} \Gamma^{U_j}(1 - f_j + F_j s_1) \prod_{j=n_2+1}^{p_2} \Gamma^{V_j}(e_j - E_j s_1)} \tag{3}$$

$$\theta_2(s_2) = \frac{\prod_{j=1}^{m_3} \Gamma^{P_j}(h_j - H_j s_2) \prod_{j=1}^{n_3} \Gamma^{Q_j}(1 - g_j + G_j s_2)}{\prod_{j=m_3+1}^{q_3} \Gamma^{P_j}(1 - h_j + H_j s_2) \prod_{j=n_3+1}^{p_3} \Gamma^{Q_j}(g_j - G_j s_2)} \tag{4}$$

Here, the variables z_1 and z_2 are non-zero real or complex numbers and an empty product is interpreted as unity. $m, n, m_1, n_1, m_2, n_2, m_3, n_3, p, q, p_1, q_1, p_2, q_2, p_3, q_3$ are all non-negative integers such that $0 \le n \le p$, $0 \le m \le q$, $0 \le n_1 \le p_1$, $0 \le m_1 \le q_1$, $0 \le n_2 \le p_2$, $0 \le m_2 \le q_2$, $0 \le n_3 \le p_3$, $0 \le m_3 \le q_3$ and $\alpha_j, \beta_j, \gamma_j, \delta_j, A_j, B_j, C_j, D_j, E_j, F_j, G_j, H_j, \xi_j, \eta_j, \xi'_j, \eta'_j, U_j, V_j, P_j, Q_j$ are all positive real numbers. $a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j$ are all complex numbers. The integration path L_1 in the complex s_1 plane runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$ such that all the poles of $\Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2)$ for $j = 1,..., m$, $\Gamma^{\eta'_j}(d_j - \delta_j s_1 + D_j s_2)$ for $j = 1,..., m_1$ and $\Gamma^{U_j}(f_j - F_j s_1)$ for $j = 1,..., m_2$ lie to the right of L_1 while all the poles of $\Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 + A_j s_2)$ for $j = 1,..., n$, $\Gamma^{\xi'_j}(1 - c_j + \gamma_j s_1 - C_j s_2)$ for $j = 1,..., n_1$ and $\Gamma^{V_j}(1 - e_j + E_j s_1)$ for $j = 1,..., n_2$ lie to the left of L_1. The integration path L_2 in the complex s_2 plane runs from $\sigma_2 - i\infty$ to $\sigma_2 + i\infty$ such that all the poles of $\Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2)$ for $j = 1,..., m$, $\Gamma^{\xi'_j}(1 - c_j + \gamma_j s_1 - C_j s_2)$ for $j = 1,..., n_1$ and $\Gamma^{P_j}(h_j - H_j s_2)$ for $j = 1,..., m_3$ lie to the right of L_2 while all the poles of $\Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 + A_j s_2)$ for $j = 1,..., n$, $\Gamma^{\eta'_j}(d_j - \delta_j s_1 + D_j s_2)$ for $j = 1,..., m_1$ and $\Gamma^{Q_j}(1 - g_j + G_j s_2)$ for $j = 1,..., n_3$ lie to the left of L_2. When the exponents of various gamma functions in equations (2), (3) and (4) are not integers, the poles of gamma functions in numerator are converted to branch points and branch cuts can be chosen that the paths of integration can be distorted for each of contours as long as that there is no coincidence of poles.

Using the results of Braaksma [1] and Rathie [9], it is easy to show that the function $I[z_1, z_2]$ defined by (1) is an analytic function of z_1 and z_2 if

$$V_1 = \sum_{j=1}^{p} \xi_j \alpha_j + \sum_{j=1}^{p_1} \xi'_j \gamma_j + \sum_{j=1}^{p_2} U_j E_j - \sum_{j=1}^{q} \eta_j \beta_j - \sum_{j=1}^{q_1} \eta'_j \delta_j - \sum_{j=1}^{q_2} V_j F_j < 0 \tag{5}$$

$$V_2 = \sum_{j=1}^{p} \xi_j A_j + \sum_{j=1}^{q_1} \eta'_j D_j + \sum_{j=1}^{p_3} P_j G_j - \sum_{j=1}^{q} \eta_j B_j - \sum_{j=1}^{p_1} \xi'_j C_j - \sum_{j=1}^{q_3} Q_j H_j < 0 \tag{6}$$

Similarly following Shantha Kumari et al. [3] and Prasad and Prasad [8], it is easy to show that the double integral defined in (1) converges absolutely if

$$|\arg z_1| < \frac{1}{2}\pi\Omega_1 \text{ and } |\arg z_2| < \frac{1}{2}\pi\Omega_2 \text{ where}$$

$$\Omega_1 = \sum_{j=1}^{n} \xi_j \alpha_j - \sum_{j=n+1}^{p} \xi_j \alpha_j + \sum_{j=1}^{m} \eta_j \beta_j - \sum_{j=m+1}^{q} \eta_j \beta_j + \sum_{j=1}^{n_1} \xi_j' \gamma_j - \sum_{j=n_1+1}^{p_1} \xi_j' \gamma_j + \sum_{j=1}^{m_1} \eta_j' \delta_j$$
$$- \sum_{m_1+1}^{q_1} \eta_j' \delta_j + \sum_{j=1}^{n_2} U_j E_j - \sum_{j=n_2+1} U_j E_j + \sum_{j=1}^{m_2} V_j F_j - \sum_{j=m_2+1}^{q_2} V_j F_j > 0 \tag{7}$$

and

$$\Omega_2 = \sum_{j=1}^{n} \xi_j A_j - \sum_{j=n+1}^{p} \xi_j A_j + \sum_{j=1}^{m} \eta_j B_j - \sum_{j=m+1}^{q} \eta_j B_j + \sum_{j=1}^{n_1} \xi_j' C_j - \sum_{j=n_1+1}^{p_1} \xi_j' C_j + \sum_{j=1}^{m_2} \eta_j' D_j$$
$$- \sum_{m_1+1}^{q_2} \eta_j' D_j + \sum_{j=1}^{n_3} P_j G_j - \sum_{j=n_3+1}^{p_3} P_j G_j + \sum_{j=1}^{m_3} Q_j H_j - \sum_{j=m_3+1}^{q_3} Q_j H_j > 0 \tag{8}$$

Now the asymptotic behavior may be establish in the following convenient form, see Braaksma [1].

$$I(z_1, z_2) = 0\,(|z_1|^{\alpha_1}, |z_2|^{\alpha_2}),\ \max\,(|z_1|^{\alpha_1}, |z_2|^{\alpha_2}) \to 0$$

$$I(z_1, z_2) = 0\,(|z_1|^{\beta_1}, |z_2|^{\beta_2}),\ \min\,(|z_1|^{\beta_1}, |z_2|^{\beta_2}) \to \infty \text{ where}$$

$$\alpha_1 = \min_{1\le j\le m_2} \Re\left[V_j\left(\frac{f_j}{F_j}\right)\right] \text{and } \alpha_2 = \min_{1\le j\le m_3} \Re\left[Q_j\left(\frac{h_j}{H_j}\right)\right]$$

$$\beta_1 = \max_{1\le j\le n_2} \Re\left[U_j\left(\frac{e_j - 1}{E_j}\right)\right] \text{ and } \beta_2 = \max_{1\le j\le n_3} \Re\left[P_j\left(\frac{g_j - 1}{G_j}\right)\right]$$

For simplicity, we shall use the following notations;

$$U = m_1, n_1 : m_2, n_2 : m_3, n_3 \tag{9}$$

$$V = p_1, q_1 : p_2, q_2 : p_3, q_3 \tag{10}$$

$$A_1 = (a_j\,; \alpha_j, A_j\,; \xi_j)_{1,\,p} \tag{11}$$

$$A_2 = (c_j; \gamma_j, C_j; \xi'_j)_{1, p_1} \tag{12}$$

$$A_3 = (e_j; E_j; U_j)_{1, p_2} \tag{13}$$

$$A_4 = (g_j; G_j; P_j)_{1, p_3} \tag{14}$$

$$B_1 = (b_j; \beta_j, B_j; \eta_j)_{1, q} \tag{15}$$

$$B_2 = (d_j; \delta_j, D_j; \eta'_j)_{1, q_1} \tag{16}$$

$$B_3 = (f_j; F_j; V_j)_{1, q_3} \tag{17}$$

$$B_4 = (h_j; H_j; Q_j)_{1, q_3} \tag{18}$$

In 1903, Mittage-Leffler [5] introduced the function $E_\alpha(y)$ in the following manner :

$$E_\alpha(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + 1)}, \text{ where } \alpha, y \in C, \Re(\alpha) > 0 \tag{19}$$

In 1905, Wiman [13] generalized the function $E_\alpha(y)$ and gave the function $E_{\alpha,\beta}(y)$ in the following manner :

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + \beta)}, \text{ where } \alpha, \beta, y \in C, \Re(\alpha) > 0, \Re(\beta) > 0 \tag{20}$$

In 1971, Prabhakar [7] generalized the function $E_{\alpha,\beta}(y)$ and gave the function $E^\lambda_{\alpha,\beta}(y)$ in the following manner :

$$E^\lambda_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(\alpha k + \beta)} \frac{y^k}{k!}, \tag{21}$$

where $\alpha, \beta, \lambda, y \in C, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0$

Saxena et al. [10] gave the multivariable analogue of multivariable Mittage-Leffler function in the following manner :

$$E^{\lambda_i}_{\mu_i,\nu_1}(y_1,\dots,y_m) = E^{\lambda_1,\dots,\lambda_m}_{\mu_1,\dots,\mu_m,\nu_1}(y_1,\dots,y_m) = \sum_{k_1,\dots,k_m=0}^{\infty} \frac{(\lambda_1)_{k_1},\dots,(\lambda_m)_{k_m}}{\Gamma\left(\nu_1 + \sum_{i=1}^{m} \mu_i k_i\right)} \frac{y_1^{k_1}}{k_1!} \cdots \frac{y_m^{k_m}}{k_m!} \tag{22}$$

where $\nu_1, \lambda_i, \mu_i \in C, \Re(\mu_j) > 0, \forall\, i = 1, 2...m.$

For the sake of convenience, let $b_m = \dfrac{(\lambda_1)_{k_1},..,(\lambda_m)_{k_m}}{\Gamma\left(\nu_1 + \sum_{i=1}^{m} \mu_i k_i\right)}$ (23)

Srivastava [11] introduced the general class of polynomials in the following manner :

$$S_N^M(y) = \sum_{k=0}^{[N/M]} \frac{(-N)_{Mk}}{k!} A_{N,k}\, y^k, N = 0, 1, 2,..... \tag{24}$$

where M is an arbitrary positive integer and the coefficients $A_{N,k}(N, k \geq 0)$ are arbitrary constants, real or complex and $(\lambda)_N$ is the pochhammer symbol.

2. Required Formulas :

The following some known integral formulas are required for our present study.

Gradshteyn and Ryzhik (see [2], p. 385, eqn. 3.666(4))

$$\int_0^{\pi} (\cos x + i \sin x \cos(a - 2\theta))^{\sigma} \cos\delta\theta\, d\theta = \frac{i^{3\delta/2}\, \pi\Gamma(\sigma+1)}{\Gamma(\sigma+\delta/2+1)} \cos(\delta a/2) P_{\sigma}^{\delta/2}(\cos x) \tag{25}$$

where $0 < x < \pi/2$

Gradshteyn and Ryzhik (see [2], p. 385, eqn. 3.666(5))

$$\int_0^{\pi} (\cos x + i \sin x \cos(a - 2\theta))^{\sigma} \sin\delta\theta\, d\theta = \frac{i^{3\delta/2}\, \pi\Gamma(\sigma+1)}{\Gamma(\sigma+\delta/2+1)} \sin(\delta a/2) P_{\sigma}^{\delta/2}(\cos x) \tag{26}$$

where $0 < x < \pi/2$

Gradshteyn and Ryzhik (see [2], p. 375, eqn. 3.633(1))

$$\int_0^{\pi} (\cos(\theta/2))^{2\sigma-1} \sin(\theta/2) \sin\delta\theta\, d\theta = \frac{2\pi\delta\Gamma(2\sigma)}{4^{\alpha}\,\Gamma(1+\sigma+\delta)\,\Gamma(1+\sigma-\delta)} \tag{27}$$

Legendre polynomial is defined by Srivastava et al. (see [12], p. 243)

$$P_{\sigma}^{\delta}(z) = \frac{1}{\Gamma(1-\delta)}\left(\frac{z+1}{z-1}\right)^{\delta/2} {}_2F_1(-\sigma, \sigma+1; \delta+1; (1-z)/2) \tag{28}$$

3. Main Integrals :

Here we are giving three finite integrals involving the Srivastava's polynomial (24), mulivariable Mittage-Leffler function (22) and the modified I-function of two variables (1).

Theorem 1 : Let $\lambda_i, \rho_i, v \in C, \eta, \eta_i, \mu_1, \mu_2 \in R^+$ for $i = 1,..., m$ such that $0 < x < \pi/2$ and $\Re(\rho_i) > 0$. Also let

$$\Re(\sigma) + \mu_1 \min_{1\le j\le m_2} \Re\left(V_j \frac{f_j}{F_j}\right) + \mu_2 \min_{1\le j\le m_3} \Re\left(Q_j \frac{h_j}{H_j}\right) > 0$$

Further let $|\arg(z_i X^{\mu_i})| < \frac{1}{2}\Omega_i\pi\,(i = 1, 2)$ with Ω_i the same as in the equations (7) and (8).

Then

$$\int_0^{\pi} (\cos x + i\sin x \cos(a-2\theta))^{\sigma} \cos\delta\theta\, S_N^M(yX^{\eta})\, E_{\rho_i,v}^{\lambda_i}(y_1X^{\eta_1},.., y_mX^{\eta_m})\, I\begin{pmatrix} z_1X^{\mu_1} \\ z_2X^{\mu_2}\end{pmatrix} d\theta$$

$$= \cos(\delta a/2)\frac{i^{3\delta/2}\pi\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4} \sum_{k=0}^{[N/M]}\sum_{k_1,..,k_m}^{\infty}\sum_{l=1}^{\infty} \frac{(-N)_{Mk}}{k!k_1!...k_m!l!\Gamma(1+\delta/2+l)}$$

$$A_{N,k}\, b_m y^k y_1^{k_1}... y_m^{k_m}\left(\frac{1-\cos x}{2}\right)^{l} I_{p+2,q+2;V}^{m+1,n+1;U}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| \begin{matrix} \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i - l; \mu_1, \mu_2; 1\right), \\ \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i + l; \mu_1, \mu_2; 1\right), \end{matrix}\right.$$

$$\left.\begin{matrix} A_1, \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i; \mu_1, \mu_2; 1\right); A_2 : A_3, A_4 \\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^{m}\eta_i k_i; \mu_1, \mu_2; 1\right); B_2 : B_3; B_4 \end{matrix}\right] \tag{29}$$

Proof : We denote the left hand side of equation (29) by Δ. Then we express the general class of polynomial and multivariable Mittage-Leffler function in the series form with the help of equations (24) and (22) and expressing the modified I-function of two variables in terms of Mellin-Barnes contour integral with the help of equation (1) and changing the order of integration, which is permissible under the given conditions, we obtain

$$\Delta = \sum_{k=0}^{[N/M]} \sum_{k_1,..,k_m}^{\infty} (-N)_{Mk} A_{N,k} b_m \frac{y^k}{k!} \frac{y_1^{k_1}}{k_1} \cdots \frac{y_m^{k_m}}{k_m} \frac{1}{(2\pi i)^r} \int_{L_1} \int_{L_2} \psi(s_1,s_2)\theta_1(s_1)\theta_2(s_2) z_1^{s1} z_2^{s2}$$

$$\times \left(\int_0^{\pi} [\cos x + i \sin x \cos(a-2\theta)]^{\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_2 s_2} \cos \delta\theta \, d\theta \right) ds_1 ds_2 \quad (30)$$

now evaluating the θ integral with the help of equation (25), we get

$$\int_0^{\pi} (\cos x + i \sin x \cos(a-2\theta))^{\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_2 s_2} \cos \delta\theta \, d\theta$$

$$= \frac{i^{3\delta/2} \pi \Gamma\left(\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_1 s_1+1\right)}{\Gamma\left(\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_1 s_1+\delta/2+1\right)} \cos(\delta a/2)\, P^{\delta/2}_{\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_1 s_1}(\cos x) \quad (31)$$

now substituting the equation (31) in the equation (30), we get

$$= \sum_{k=0}^{[N/M]} \sum_{k_1,..,k_m}^{\infty} (-N)_{Mk} A_{N,k} b_m \frac{y^k}{k!} \frac{y_1^{k_1}}{k_1} \cdots \frac{y_m^{k_m}}{k_m} \frac{1}{(2\pi i)^r} \int_{L_1} \int_{L_2} \psi(s_1,s_2)\theta_1(s_1)\theta_2(s_2) z_1^{s_1} z_2^{s_2}$$

$$\times \frac{i^{3\delta/2} \pi \Gamma\left(\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_1 s_1+1\right)}{\Gamma\left(\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_1 s_1+\delta/2+1\right)} \cos(\delta a/2)\, P^{\delta/2}_{\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_1 s_1} (\cos x)\, ds_1 ds_2 \quad (32)$$

now using equation (28), we get

$$P^{\delta/2}_{\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_1+\mu_1 s_1+\mu_1 s_1}(\cos x) = \frac{1}{\Gamma(1-\delta/2)} \left(\frac{\cos x+1}{\cos x-1} \right)^{\delta/4}$$

$$_2F_1\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-\mu_1 s_1-\mu_1 s_1, \sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_1 s_1+1;\delta/2+1;(1-\cos x)/2\right) \tag{33}$$

Simplifying the above equation (33), we get

$$P^{\delta/2}_{\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_1+\mu_1 s_1+\mu_1 s_1}(\cos x)=\frac{1}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4}$$

$$\sum_{l=1}^{\infty}\frac{\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-\mu_1 s_1-\mu_1 s_1\right)_l}{(\delta/2+1)_l}\times\left(\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_1+\mu_1 s_1+\mu_1 s_1+1\right)_l\frac{1}{l!}\left(\frac{1-\cos x}{2}\right)^l \tag{34}$$

now using the result $\Gamma(\sigma+l)=(\sigma)_l\,\Gamma(\sigma)$, the above equation (34) reduces to

$$P^{\delta/2}_{\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_1+\mu_1 s_1+\mu_1 s_1}(\cos x)=\frac{1}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4}$$

$$\sum_{l=1}^{\infty}\frac{\Gamma\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i+l-\mu_1 s_1-\mu_1 s_1\right)}{\Gamma\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-\mu_1 s_1-\mu_1 s_1\right)}$$

$$\times\frac{\Gamma(1+\delta/2)\,\Gamma\left(1+\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+l+\mu_1 s_1+\mu_1 s_1\right)}{\Gamma(1+\delta/2+l)\,\Gamma\left(1+\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\mu_1 s_1+\mu_1 s_1\right)}\frac{1}{l!}\left(\frac{1-\cos x}{2}\right)^l \tag{35}$$

Now putting the equation (35) in equation (32), we get

$$=\cos(\delta a/2)\frac{i^{3\delta/2}\pi\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4}\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_m}^{\infty}\sum_{l=0}^{\infty}\frac{(-N)_{Mk}}{\Gamma(1+\delta/2+l)}A_{N,k}b_m$$

$$\frac{y^k}{k!}\frac{y_1^{k_1}}{k_1}\cdots\frac{y_m^{k_m}}{k_m}\times\frac{1}{l!}\left(\frac{1-\cos x}{2}\right)^l\int_{L_1}\int_{L_2}\psi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)$$

$$\frac{\Gamma\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i+l-\mu_1 s_1-\mu_1 s_1\right)}{\Gamma\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-\mu_1 s_1-\mu_1 s_1\right)}$$

$$\times \frac{\Gamma\left(1+\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+l+\mu_1 s_1+\mu_1 s_1\right)}{\Gamma\left(1+\sigma+\eta k+\sum_{i=1}^{m}\eta_i k_i+\delta/2+\mu_1 s_1+\mu_1 s_1\right)} z_1^{s_1} z_2^{s_2} ds_1 ds_2 \tag{36}$$

now interpreting the equation (36) with the help of equation (1) in terms of Mellin-Barnes contour integral, we get required result (29).

Special Case :

By taking $m_1, n_1, p_1, q_1 = 0$ theorem 1 reduces to the following corollary for the generalized I-function of two variables.

Corollary 1 : Let $\lambda_i, \rho_i, v \in C, \eta, \eta_i, \mu_1, \mu_2 \in R^+$ for $i = 1,..., m$ such that $0 < x < \pi/2$ and $\Re(\rho_i) > 0$. Also let

$$\Re(\sigma) + \mu_1 \min_{1\le j\le m_2} \Re\left(V_j \frac{f_j}{F_j}\right) + \mu_2 \min_{1\le j\le m_3} \Re\left(Q_j \frac{h_j}{H_j}\right) > 0$$

Further let $|\arg(z_i X^{\mu_i})| < \frac{1}{2}\Omega_i \pi\,(i = 1, 2)$ with Ω_i the same as in the equations (37) and (38) given below

$$\Omega_1 = \sum_{j=1}^{n}\xi_j\alpha_j - \sum_{j=n+1}^{p}\xi_j\alpha_j - \sum_{j=1}^{q}\eta_j\beta_j + \sum_{j=1}^{n_2}U_jE_j - \sum_{j=n_2+1}^{p_2}U_jE_j + \sum_{j=1}^{m_2}V_jF_j - \sum_{j=m_2+1}^{q_2}V_jF_j > 0 \tag{37}$$

and

$$\Omega_2 = \sum_{j=1}^{n}\xi_jA_j - \sum_{j=n+1}^{p}\xi_jA_j - \sum_{j=1}^{q}\eta_jB_j + \sum_{j=1}^{n_3}P_jG_j - \sum_{j=n_3+1}^{p_3}P_jG_j + \sum_{j=1}^{m_3}Q_jH_j - \sum_{j=m_3+1}^{q_3}Q_jH_j > 0 \tag{38}$$

Then

$$\int_0^{\pi}(\cos x + i\sin x\cos(a-2\theta))^{\sigma}\cos\delta\theta\, S_N^M(yX^{\eta})\, E_{\rho_i,v}^{\lambda_i}(y_1X^{\eta_1},..., y_mX^{\eta_m})\, I\begin{pmatrix} z_1X^{\mu_1} \\ z_2X^{\mu_2}\end{pmatrix} d\theta$$

$$= \cos(\delta a/2)\frac{i^{3\delta/2}\pi\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4}\sum_{k=0}^{[N/M]}\sum_{k_1,...,k_m}^{\infty}\sum_{l=1}^{\infty}\frac{(-N)_{Mk}}{k!k_1!...k_m!l!\Gamma(1+\delta/2+l)}$$

$$A_{N,k} b_m y^k y_1^{k_1} \cdots y_m^{k_m} \left(\frac{1-\cos x}{2}\right)^l I^{m+1,n+1:m_2,n_2;m_3,n_3}_{p+2,q+2:p_2,q_2,p_3,q_3}\left[\begin{array}{c|c} z_1 & \left(-\sigma-\eta k-\sum_{i=1}^m \eta_i k_i - l;\mu_1,\mu_2;1\right), \\ z_2 & \left(-\sigma-\eta k-\sum_{i=1}^m \eta_i k_i + l;\mu_1,\mu_2;1\right), \end{array}\right.$$

$$\left.\begin{array}{c} A_1, \left(-\sigma-\eta k-\sum_{i=1}^m \eta_i k_i;\mu_1,\mu_2;1\right): A_3; A_4 \\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^m \eta_i k_i;\mu_1,\mu_2;1\right): B_3; B_4 \end{array}\right] \tag{39}$$

Theorem 2 : Let $\lambda_i, \rho_i, v \in C, \eta, \eta_i, \mu_1, \mu_2 \in R^+$ for $i = 1,..., m$ such that $0 < x < \pi/2$ and $\Re(\rho_i) > 0$. Also let

$$\Re(\sigma) + \mu_1 \min_{1\le j\le m_2} \Re\left(V_j \frac{f_j}{F_j}\right) + \mu_2 \min_{1\le j\le m_3} \Re\left(Q_j \frac{h_j}{H_j}\right) > 0$$

Further let $|\arg(z_i X^{\mu_i})| < \frac{1}{2}\Omega_i \pi\ (i = 1, 2)$ with Ω_i the same as in the equations (7) and (8).

Then

$$\int_0^\pi [\cos x + i \sin x \cos(a-2\theta)]^\sigma \sin\delta\theta\, S_N^M(yX^\eta)\, E^{\lambda_i}_{\rho_i,v}(y_1 X^{\eta_1},..., y_m X^{\eta_m})\, I\begin{pmatrix} z_1 X^{\mu_1} \\ z_2 X^{\mu_2}\end{pmatrix} d\theta$$

$$= \sin(\delta a/2)\frac{i^{3\delta/2}\pi\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4} \sum_{k=0}^{[N/M]} \sum_{k_1,...,k_m}^{\infty} \sum_{l=1}^{\infty} \frac{(-N)_{Mk}}{k! k_1! \cdots k_m! l!\, \Gamma(1+\delta/2+l)}$$

$$A_{N,k} b_m y^k y_1^{k_1} \cdots y_m^{k_m} \left(\frac{1-\cos x}{2}\right)^l I^{m+1,n+1;U}_{p+2,q+2;V}\left[\begin{array}{c|c} z_1 & \left(-\sigma-\eta k-\sum_{i=1}^m \eta_i k_i - l;\mu_1,\mu_2;1\right) \\ z_2 & \left(-\sigma-\eta k-\sum_{i=1}^m \eta_i k_i + l;\mu_1,\mu_2;1\right) \end{array}\right.$$

$$\left.\begin{array}{c} A_1, \left(-\sigma-\eta k-\sum_{i=1}^m \eta_i k_i;\mu_1,\mu_2;1\right); A_2: A_3; A_4 \\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^m \eta_i k_i;\mu_1,\mu_2;1\right); B_2: B_3; B_4 \end{array}\right] \tag{40}$$

Proof : We omit the proof. It is similar to the proof of previous theorem.

Special Case :

By taking $m_1, n_1, p_1, q_1 = 0$ theorem 2 reduces to the following corollary for the generalized I-function of two variables.

Corollary 2 : Let $\lambda_i, \rho_i, v \in C, \eta, \eta_i, \mu_1, \mu_2 \in R^+$ for $i = 1,..., m$ such that $0 < x < \pi/2$ and $\Re(\rho_i) > 0$. Also let

$$\Re(\sigma) + \mu_1 \min_{1\le j\le m_2} \Re\left(V_j \frac{f_j}{F_j}\right) + \mu_2 \min_{1\le j\le m_3} \Re\left(Q_j \frac{h_j}{H_j}\right) > 0.$$

Further let $|\arg(z_i X^{\mu_i})| < \frac{1}{2}\Omega_i \pi\,(i = 1, 2)$ with Ω_i the same as in the equations (37) and (38). Then

$$\int_0^\pi (\cos x + i\sin x \cos(a-2\theta))^\sigma \sin\delta\theta\, S_N^M(yX^\eta)\, E_{\rho_i,v}^{\lambda_i}(y_1X^{\eta_1},..., y_mX^{\eta_m})\, I\begin{pmatrix} z_1X^{\mu_1} \\ z_2X^{\mu_2}\end{pmatrix} d\theta$$

$$= \sin(\delta a/2)\frac{i^{3\delta/2}\pi\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4} \sum_{k=0}^{[N/M]}\sum_{k_1,...,k_m}^{\infty}\sum_{l=1}^{\infty}\frac{(-N)_{Mk}}{k!\,k_1!...k_m!\,l!\,\Gamma(1+\delta/2+l)}$$

$$A_{N,k}b_{m,}y^k y_1^{k_1}... y_m^{k_m}\left(\frac{1-\cos x}{2}\right)^l I^{m+1,n+1:m_2,n_2;m_3,n_3}_{p+2,q+2:p_2,q_2,p_3,q_3}\left[\begin{matrix} z_1 \\ z_2\end{matrix}\middle|\begin{matrix}\left(-\sigma-\eta k-\sum_{i=1}^m \eta_ik_i-l;\mu_1,\mu_2;1\right), \\ \left(-\sigma-\eta k-\sum_{i=1}^m \eta_ik_i+l;\mu_1,\mu_2;1\right),\end{matrix}\right.$$

$$\left.\begin{matrix} A_1, \left(-\sigma-\eta k-\sum_{i=1}^m \eta_ik_i;\mu_1,\mu_2;1\right): A_3; A_4 \\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^m \eta_ik_i;\mu_1,\mu_2;1\right): B_3; B_4\end{matrix}\right] \tag{41}$$

Theorem 3 : Let $\lambda_i, \rho_i, v \in C, \eta, \eta_i, \mu_1, \mu_2 \in R^+$ for $i = 1,..., m$ such that $0 < x < \pi/2$ and $\Re(\rho_i) > 0$. Also let

$$\Re(2\sigma) + 2\mu_1 \min_{1\le j\le m_2} \Re\left(V_j \frac{f_j}{F_j}\right) + 2\mu_2 \min_{1\le j\le m_3} \Re\left(Q_j \frac{h_j}{H_j}\right) > 0$$

Further let $\left|\arg\left(z_i(\cos\theta/2)^{2\mu_i}\right)\right| < \frac{1}{2}\Omega_i\pi\ (i = 1, 2)$ with Ω_i the same as in the equations (7) and (8). Then

$$\int_0^{\pi} (\cos(\theta/2))^{2\sigma-1} \sin(\theta/2) \sin \delta\theta \, S_N^M(y(\cos\theta/2)^{2\eta}) \, E_{\rho_i,\nu}^{\lambda_i} (y_1(\cos\theta/2)^{2\eta_1},...,$$

$$y_m(\cos\theta/2)^{2\eta_m}) \times I \begin{pmatrix} z_1(\cos\theta/2)^{2\mu_1} \\ z_2(\cos\theta/2)^{2\mu_2} \end{pmatrix} d\theta$$

$$= \frac{2\delta\pi}{4^{\sigma}} \sum_{k=0}^{[N/M]} \sum_{k_1,...,k_m}^{\infty} \frac{(-N)_{Mk}}{k!\, k_1! \cdots k_m!} A_{N,k}\, b_m \left(\frac{y}{4^{\eta}}\right)^k \left(\frac{y_1}{4^{\eta_1}}\right)^{k_1} \cdots \left(\frac{y_m}{4^{\eta_m}}\right)^{k_m} \times I_{p+1,q+2;V}^{m,n+1;U} \left[\begin{matrix} z_1 4^{-\mu_1} \\ z_2 4^{-\mu_2} \end{matrix} \middle| \begin{matrix} \left(1-2\sigma-2\eta k-2\sum_{i=1}^m \eta_i k_i; 2\mu_1, 2\mu_2; 1\right), A_1; A_1 : A_3; A_4 \\ B_1, \left(-\sigma-\eta k-\sum_{i=1}^m \eta_i k_i - \delta; \mu_1, \mu_2; 1\right), \left(-\sigma-\eta k-\sum_{i=1}^m \eta_i k_i + \delta; \mu_1, \mu_2; 1\right); B_2 : B_3; B_4 \end{matrix} \right] \quad (42)$$

Proof : We omit the proof. It is similar to the proof of theorem 1.

Special Case :

By taking $m_1, n_1, p_1, q_1 = 0$ theorem 3 reduces to the following corollary for the generalized I-function of two variables.

Corollary 3 : Let $\lambda_i, \rho_i, \nu \in C, \eta, \eta_i, \mu_1, \mu_2 \in R^+$ for $i = 1,..., m$ such that $0 < x < \pi/2$ and $\Re(\rho_i) > 0$. Also let

$$\Re(2\sigma) + 2\mu_1 \min_{1 \le j \le m_2} \Re\left(V_j \frac{f_j}{F_j}\right) + 2\mu_2 \min_{1 \le j \le m_3} \Re\left(Q_j \frac{h_j}{H_j}\right) > 0$$

Further let $|\arg\left(z_i (\cos\theta/2)^{2\mu_i}\right)| < \frac{1}{2}\Omega_i \pi$ $(i = 1, 2)$ with Ω_i the same as in the equations (37) and (38). Then

$$\int_0^{\pi} (\cos(\theta/2))^{2\sigma-1} \sin(\theta/2) \sin \delta\theta \, S_N^M(y(\cos\theta/2)^{2\eta}) \, E_{\rho_i,\nu}^{\lambda_i} (y_1(\cos\theta/2)^{2\eta_1},...,$$

$$y_m(\cos\theta/2)^{2\eta_m}) \times I \begin{pmatrix} z_1(\cos\theta/2)^{2\mu_1} \\ z_2(\cos\theta/2)^{2\mu_2} \end{pmatrix} d\theta = \frac{2\delta\pi}{4^{\sigma}}$$

$$\sum_{k=0}^{[N/M]} \sum_{k_1,...,k_m}^{\infty} \frac{(-N)_{Mk}}{k!\, k_1! \cdots k_m!} A_{N,k}\, b_m \left(\frac{y}{4^{\eta}}\right)^k \left(\frac{y_1}{4^{\eta_1}}\right)^{k_1} \cdots \left(\frac{y_m}{4^{\eta_m}}\right)^{k_m} \times I_{p+1,q+2:p_2,q_2;p_3,q_3}^{m,n+1:m_2;n_2;m_3;n_3} \left[\begin{matrix} z_1 4^{-\mu_1} \\ z_2 4^{-\mu_2} \end{matrix} \middle| \right.$$

$$\left.\begin{matrix} \left(1-2\sigma-2\eta k-2\sum_{i=1}^{m}\eta_i k_i;2\mu_1,2\mu_2;1\right),A_1;A_3;A_4 \\ B_1,\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-\delta;\mu_1,\mu_2;1\right),\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i+\delta;\mu_1,\mu_2;1\right):B_3;B_4 \end{matrix}\right] \tag{43}$$

4. Fourier series :

In this section, we shall obtain three Fourier series with the help of integrals obtained in the theorems of previous section.

First Fourier series :

$$(\cos x+i\sin x\cos(a-2\theta))^{\sigma}S_N^M(yX^{\eta})E_{\rho_i,v}^{\lambda_i}(y_1X^{\eta_1},..,y_mX^{\eta_m})\,I\begin{pmatrix} z_1X^{\mu_1} \\ z_2X^{\mu_2}\end{pmatrix}$$

$$=\sum_{k=0}^{[N/M]}\sum_{k_1,..,k_m}^{\infty}\sum_{l=1}^{\infty}\frac{(-N)_{Mk}}{k!\,k_1!\ldots k_m!\,l!\,\Gamma(1+l)}A_{N,k}\,b_m\,y^k y_1^{k_1}\ldots y_m^{k_m}\left(\frac{1-\cos x}{2}\right)^l$$

$$\times I_{p+2,q+2;V}^{m+1,n+1;U}\left[\begin{matrix} z_1 \\ z_2\end{matrix}\middle|\begin{matrix} \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-l;\mu_1,\mu_2;1\right), \\ \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i+l;\mu_1,\mu_2;1\right),\end{matrix}\right.$$

$$\left.\begin{matrix} A_1,\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right);A_2:A_3;A_4 \\ B_1,\left(-\sigma-\eta k-\delta/2-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right);B_2:B_3;B_4\end{matrix}\right]$$

$$+\sum_{\delta=1}^{\infty}\sum_{k=0}^{[N/M]}\sum_{k_1,..,k_m}^{\infty}\sum_{l=1}^{\infty}\frac{(-N)_{Mk}2i^{3\delta/2}\cos(\delta a/2)}{k!k_1!\cdots k_m!\,l!\,\Gamma(1+\delta/2+l)}\frac{\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4}$$

$$A_{N,k}b_m y^k y_1^{k_1}\ldots y_m^{k_m}\left(\frac{1-\cos x}{2}\right)^l I_{p+2,q+2;V}^{m+1,n+1;U}\left[\begin{matrix} z_1 \\ z_2\end{matrix}\middle|\begin{matrix} \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-l;\mu_1,\mu_2;1\right), \\ \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i+l;\mu_1,\mu_2;1\right),\end{matrix}\right.$$

$$\left.\begin{matrix} A_1,\left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right);A_2:A_3;A_4 \\ B_1,\left(-\sigma-\eta k-\delta/2-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right);B_2:B_3;B_4\end{matrix}\right]\cos\delta\theta \tag{44}$$

Proof : To prove (39), let

$$f(\theta) = (\cos x + i \sin x \cos(a-2\theta))^{\sigma} S_N^M(yX^{\eta}) E_{\rho_i,v}^{\lambda_i}(y_1 X^{\eta_1},\dots,y_m X^{\eta_m})\, I\begin{pmatrix} z_1 X^{\mu_1} \\ z_2 X^{\mu_2} \end{pmatrix}$$

$$(1/2)\, C_0 + \sum_{\delta=1}^{\infty} C_{\delta} \cos \delta\theta \qquad 0 \le \theta \le \pi \tag{45}$$

Multiplying both sides of equation (40) by $\cos \lambda\theta$ and integrating from 0 to π and using equation (29) and orthogonal property of cosine functions, we get

$$C_{\lambda}\pi/2 = \cos(\lambda a/2) \frac{i^{3\lambda/2}\pi\Gamma(1+\lambda/2)}{\Gamma(1-\lambda/2)} \left(\frac{\cos x+1}{\cos x-1}\right)^{\lambda/4} \sum_{k=0}^{[N/M]} \sum_{k_1,\dots,k_m}^{\infty} \sum_{l=1}^{\infty}$$

$$\frac{(-N)_{Mk}}{k!k_1!\cdots k_m! l!\Gamma(1+\lambda/2+l)} A_{N,k}\, b_m y^k y_1^{k_1}\cdots y_m^{k_m} \left(\frac{1-\cos x}{2}\right)^{l} I_{p+2,q+2;V}^{m+1,n+1;U}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\right|$$

$$\left.\begin{matrix} \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i - l;\mu_1,\mu_2;1\right),\ A_1, \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right); A_2 : A_3 ; A_4 \\ \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i + l;\mu_1,\mu_2;1\right),\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right); B_2 : B_3 ; B_4 \end{matrix}\right] \tag{46}$$

Now using equation (40) and (41), we get the result (39).

Special Case :

By taking $m_1, n_1, p_1, q_1 = 0$ Fourier series 1 reduces to the following Fourier series for the generalized *I*-function of two variables.

Corollary 4 :

$$(\cos x + i \sin x \cos(a-2\theta))^{\sigma} S_N^M(yX^{\eta}) E_{\rho_i,v}^{\lambda_i}(y_1 X^{\eta_1},\dots,y_m X^{\eta_m})\, I\begin{pmatrix} z_1 X^{\mu_1} \\ z_2 X^{\mu_2} \end{pmatrix}$$

$$= \sum_{k=0}^{[N/M]} \sum_{k_1,\dots,k_m}^{\infty} \sum_{l=1}^{\infty} \frac{(-N)_{Mk}}{k!k_1!\dots k_m! l!\Gamma(1+l)} A_{N,k}\, b_m y^k y_1^{k_1} \dots y_m^{k_m} \left(\frac{1-\cos x}{2}\right)^{l}$$

$$\times I^{m+1,n+1:m_2,n_2;m_3,n_3}_{p+2,q+2:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i - l;\mu_1,\mu_2;1\right), \\ z_2 & \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i + l;\mu_1,\mu_2;1\right), \end{array}\right.$$

$$\left.\begin{array}{l} A_1, \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right): A_3; A_4 \\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right): B_3; B_4 \end{array}\right]$$

$$+\sum_{\delta=1}^{\infty}\sum_{k=0}^{[N/M]}\sum_{k_1,\dots,k_m}^{\infty}\sum_{l=1}^{\infty}\frac{(-N)_{Mk}2i^{3\delta/2}\cos(\delta a/2)}{k!k_1!\cdots k_m!\,l!\,\Gamma(1+\delta/2+l)}\frac{\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4}A_{N,k}b_m$$

$$y^k y_1^{k_1}\cdots y_m^{k_m}\left(\frac{1-\cos x}{2}\right)^l I^{m+1,n+1:m_2,n_2;m_3,n_3}_{p+2,q+2:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i - l;\mu_1,\mu_2;1\right), \\ z_2 & \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i + l;\mu_1,\mu_2;1\right), \end{array}\right.$$

$$\left.\begin{array}{l} A_1, \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right): A_3; A_4 \\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right): B_3; B_4 \end{array}\right]\cos\delta\theta \qquad (47)$$

Second Fourier Series :

$$(\cos x+i\sin x\cos(a-2\theta))^{\sigma}S_N^M(yX^{\eta})E^{\lambda_i}_{\rho_i,\nu}(y_1X^{\eta_1},\dots,y_mX^{\eta_m})\,I\begin{pmatrix} z_1X^{\mu_1} \\ z_2X^{\mu_2}\end{pmatrix}$$

$$=\sum_{\delta=1}^{\infty}\sum_{k=0}^{[N/M]}\sum_{k_1,\dots,k_m}^{\infty}\sum_{l=1}^{\infty}\frac{(-N)_{Mk}2i^{3\delta/2}\sin(\delta a/2)}{k!k_1!\cdots k_m!\,l!\,\Gamma(1+\delta/2+l)}\frac{\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4}$$

$$A_{N,k}b_m y^k y_1^{k_1}\cdots y_m^{k_m}\left(\frac{1-\cos x}{2}\right)^l I^{m+1,n+1;U}_{p+2,q+2;V}\left[\begin{array}{c|l} z_1 & \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i - l;\mu_1,\mu_2;1\right), \\ z_2 & \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i + l;\mu_1,\mu_2;1\right), \end{array}\right.$$

$$\left.\begin{array}{l} A_1, \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right); A_2: A_3; A_4 \\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right); B_2: B_3; B_4 \end{array}\right]\sin\delta\theta \qquad (48)$$

Special Case :

By taking $m_1, n_1, p_1, q_1 = 0$ Fourier series 2 reduces to the following Fourier series for the generalized I-function of two variables.

Corollary 5 :

$$(\cos x + i\sin x\cos(a-2\theta))^{\sigma}\, S_N^M(yX^{\eta})\, E_{\rho_i,\nu}^{\lambda_i}(y_1X^{\eta_1},..., y_mX^{\eta_m})\, I\begin{pmatrix} z_1X^{\mu_1} \\ z_2X^{\mu_2}\end{pmatrix}$$

$$= \sum_{\delta=1}^{\infty}\sum_{k=0}^{[N/M]}\sum_{k_1,...,k_m}^{\infty}\sum_{l=1}^{\infty} \frac{(-N)_{Mk}\, 2i^{3\delta/2}\sin(\delta a/2)}{k!k_1!\cdots k_m!\, l!\,\Gamma(1+\delta/2+l)}\,\frac{\Gamma(1+\delta/2)}{\Gamma(1-\delta/2)}\left(\frac{\cos x+1}{\cos x-1}\right)^{\delta/4} A_{N,k}\, b_m$$

$$y^k y_1^{k_1}\cdots y_m^{k_m}\left(\frac{1-\cos x}{2}\right)^{l} I_{p+2,q+2:p_2,q_2;p_3,q_3}^{m+1,n+1:m_2,n_2;m_3,n_3}\left[\begin{matrix} z_1 \\ z_2\end{matrix}\left|\begin{matrix} \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i - l;\mu_1,\mu_2;1\right), \\ \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i + l;\mu_1,\mu_2;1\right),\end{matrix}\right.\right.$$

$$\left.\begin{matrix} A_1, \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right): A_3; A_4 \\ B_1, \left(-\sigma-\eta k-\delta/2-\sum_{i=1}^{m}\eta_i k_i;\mu_1,\mu_2;1\right): B_3; B_4\end{matrix}\right]\sin\delta\theta \qquad (49)$$

Third Fourier Series :

$$(\cos(\theta/2))^{2\sigma-1}\sin(\theta/2)\, S_N^M(y(\cos\theta/2)^{2\eta})\, E_{\rho_i,\nu}^{\lambda_i}(y_1(\cos\theta/2)^{2\eta_1},...., y_m(\cos\theta/2))$$

$$I\begin{pmatrix} z_1(\cos\theta/2)^{2\mu_1} \\ z_2(\cos\theta/2)^{2\mu_2}\end{pmatrix} = \sum_{\delta=1}^{\infty}\sum_{k=0}^{[N/M]}\sum_{k_1,...,k_m}^{\infty}\frac{4\delta}{4^{\sigma}}\frac{(-N)_{Mk}}{k!k_1!\cdots k_m!}A_{N,k}\, b_m\left(\frac{y}{4^{\eta}}\right)^{k}\left(\frac{y_1}{4^{\eta_1}}\right)^{k_1}\cdots\left(\frac{y_m}{4^{\eta_m}}\right)^{k_m}$$

$$\times I_{p+1,q+2;V}^{m,n+1;U}\left[\begin{matrix} z_1 4^{-\mu_1} \\ z_2 4^{-\mu_2}\end{matrix}\left|\begin{matrix} \left(1-2\sigma-2\eta k-2\sum_{i=1}^{m}\eta_i k_i;2\mu_1,2\mu_2;1\right), A_1; \\ B_1, \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-\delta;\mu_1,\mu_2;1\right),\end{matrix}\right.\right.$$

$$\left.\begin{matrix} A_2: A_3; A_4 \\ \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i+\delta;\mu_1,\mu_2;1\right); B_2: B_3; B_4\end{matrix}\right]\sin\delta\theta \qquad (50)$$

Special Case :

By taking $m_1, n_1, p_1, q_1 = 0$ Fourier series 3 reduces to the following Fourier series for the generalized I-function of two variables.

Corollary 6 :

$$(\cos(\theta/2))^{2\sigma-1}\sin(\theta/2)\, S_N^M(y(\cos\theta/2)^{2\eta})\, E_{\rho_i,\nu}^{\lambda_i}(y_1(\cos\theta/2)^{2\eta_1},\ldots, y_m(\cos\theta/2))$$

$$I\begin{pmatrix} z_1(\cos\theta/2)^{2\mu_1} \\ z_2(\cos\theta/2)^{2\mu_2}\end{pmatrix} = \sum_{\delta=1}^{\infty}\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_m}^{\infty} \frac{4\delta}{4^{\sigma}}\frac{(-N)_{Mk}}{k!\,k_1!\cdots k_m!} A_{N,k}\, b_m \left(\frac{y}{4^{\eta}}\right)^{k}\left(\frac{y_1}{4^{\eta_1}}\right)^{k_1}\cdots\left(\frac{y_m}{4^{\eta_m}}\right)^{k_m}$$

$$\times I_{p+1,q+2:p_2,q_2;p_3,q_3}^{m,n+1:m_2,n_2;m_3,n_3}\left[\begin{matrix} z_1 4^{-\mu_1} \\ z_2 4^{-\mu_2}\end{matrix}\middle| \begin{matrix}\left(1-2\sigma-2\eta k-2\sum_{i=1}^{m}\eta_i k_i ; 2\mu_1, 2\mu_2 ; 1\right), A_1; A_3;A_4 \\ B_1, \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i-\delta ; \mu_1, \mu_2 ; 1\right), \left(-\sigma-\eta k-\sum_{i=1}^{m}\eta_i k_i+\delta ; \mu_1, \mu_2 ; 1\right); B_3; B_4\end{matrix}\right] \sin\delta\theta \qquad (51)$$

5. Conclusion :

In this paper, we have derived three integrals, three Fourier series and several special cases involving general class of polynomial of one variable, multivariable Mittage-Leffler function and modified I-function of two variables. The importance of general class of polynomial, multivariable Mittage-Leffler function and modified I-function lies in the fact that, several simpler special functions are the particular cases of these functions, so that each of the results derived in this paper for these functions becomes a master result from which a large number of relations for other simpler special functions can be derived by the suitable specializing the parameters of these functions.

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